Probability distribution for large N :

From last lecture note

$$W(n_1) = \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} \quad \text{--- (1)}$$

$W(n_1) \rightarrow$ Prob. of taking n_1 steps to the right & n_2 steps to the left where $n_2 = N - n_1$

For large N we can approximate W to be a continuous function of variable n_1. Since

$$|W(n_1+1) - W(n_1)| << W(n_1)$$

· Here we consider regions near maximum of W where n_1 is also large

The maximum of W at location $n_1 = \bar{n}_1$ is determined by the condition

$$\left.\begin{aligned} \frac{dW}{dn_1} &= 0 \\ \text{or} \quad \frac{d\ln W}{dn_1} &= 0 \end{aligned}\right\} \quad \text{--- (2)}$$

where derivatives are evaluated at $n_1 = \bar{n}_1$

Let us take $n_1 \equiv \bar{n}_1 + \eta$ --- (3)

Eq. (3) is used to obtain the behavior of W near its maximum.

$$W(n_1) \equiv W(\bar{n}_1 + \eta) \quad \text{--- (4)}$$

Taylor series of a function $f(x+h) \longrightarrow$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots \quad \text{--- (5)}$$

Now Taylor expansion of ~~W($\bar{n}_1$)~~ $\ln W(n_1)$ is obtained as

$$\ln W(n_1) = \ln W(\bar{n}_1) + \eta \left.\frac{d \ln W(n_1)}{dn_1}\right|_{n_1=\bar{n}_1} + \frac{\eta^2}{2}\left.\frac{d^2 \ln W(n_1)}{dn_1^2}\right|_{n_1=\bar{n}_1} + \frac{\eta^3}{6}\left.\frac{d^3 \ln W(n_1)}{dn_1^3}\right|_{n_1=\bar{n}_1} + \cdots$$

or $$\ln W(n_1) = \ln W(\bar{n}_1) + \eta B_1 + \frac{\eta^2}{2} B_2 + \frac{\eta^3}{6} B_3 + \cdots \quad (6)$$

where we define $B_k = \left.\dfrac{d^k \ln W}{d n_1^k}\right|_{n_1=\bar{n}_1}$ (7)

From (2) $$\left.\frac{d \ln W(n_1)}{dn_1}\right|_{n_1=\bar{n}_1} = 0$$

$$\Rightarrow B_1 = 0$$

Take $W(\bar{n}_1) = \tilde{W}$

we write $$\ln W(n_1) = \ln \tilde{W} + \cancel{\eta B_1}^{\,0} + \frac{\eta^2}{2} B_2 + \frac{\eta^3}{6} B_3 + \cdots$$

or $$\ln\left(\frac{W(n_1)}{\tilde{W}}\right) = \frac{\eta^2}{2} B_2 + \frac{\eta^3}{6} B_3 + \cdots$$

or $$W(n_1) = \tilde{W}\, e^{\left(\frac{1}{2} B_2 \eta^2 + \frac{1}{6} B_3 \eta^3 + \cdots\right.}$$

Since W is maximum $\Rightarrow$ second derivative $$\frac{d^2 \ln W}{dn_1^2} < 0$$

we write $B_2 = -|B_2|$

Thus $$W(n_1) = \tilde{W}\, e^{-\frac{1}{2}|B_2|\eta^2 + \frac{1}{6} B_3 \eta^3 + \cdots}$$

For η to be very small, we can neglect higher order terms in the Taylor expansion and write

$$\boxed{W(n_1) = \tilde{W} e^{-\frac{1}{2}|B_2|\eta^2}} \quad \text{—— (8)}$$

Next, we discuss expansion (7) in ~~discrete~~ more elaborate form using eqn (1)

~~$\ln W(n_1) = \ln N!$~~

$$W(n_1) = \frac{N!}{n_1!\,(N-n_1)!} p^{n_1} q^{N-n_1}$$

Taking log on both sides

$$\ln W(n_1) = \ln\left(\frac{N!}{n_1!\,(N-n_1)!}\right) + \ln(p^{n_1} q^{N-n_1})$$

$$= \ln N! - \ln n_1! - \ln(N-n_1)! + \ln p^{n_1} + \ln q^{N-n_1}$$

$$\text{or } \ln W(n_1) = \ln N! - \ln n_1! - \ln(N-n_1)! + n_1 \ln p + (N-n_1)\ln q \quad \text{—— (9)}$$

for $n \gg 1$, $\ln n! \rightarrow$ ~~[illegible]~~ continuous

We can write

$$\frac{d \ln n!}{dn} \approx \frac{\ln(n+1)! - \ln n!}{1} \quad \text{—— (10)}$$

Since $\ln n!$ changes only by a small fraction if n changed by a small number.

We can write (10) as

$$\frac{d \ln n!}{dn} = \ln\left[\frac{(n+1)!}{n!}\right]$$

or $\frac{d \ln n!}{dn} = \ln(n+1)$

For $n \gg 1$, $\frac{d \ln n!}{dn} \approx \ln n$ — (11)

Thus from (1) & (9)

$$\frac{d \ln W(n_1)}{dn_1} = - \frac{d \ln n_1!}{dn_1} - \frac{d \ln (N-n_1)!}{dn_1} + \ln p - \ln q$$

$$\boxed{\frac{d \ln W(n_1)}{dn_1} = - \ln n_1 + \ln (N-n_1) + \ln p - \ln q}$$ — (12)

From Maximum Condition $\left.\frac{d \ln W(n_1)}{dn_1}\right|_{n=\bar{n}_1} = 0$

we can write

$$0 = - \ln \bar{n}_1 + \ln (N-\bar{n}_1) + \ln p - \ln q$$

or $\ln\left(\frac{N-\bar{n}_1}{\bar{n}_1}\right) + \ln\left(\frac{p}{q}\right) = 0$

or $\ln\left[\left\{\frac{N-\bar{n}_1}{\bar{n}_1}\right\}\frac{p}{q}\right] = 0$

~~or $(N-\bar{n}_1)p = n_1 q$~~

~~Since $p+q=1$~~

or $(N-\bar{n}_1)p = \bar{n}_1 q$

Since $p+q=1 \Rightarrow \boxed{\bar{n}_1 = Np}$ — (13)

Next, we evaluate the second derivative of (12)

$$\frac{d^2 \ln W(n_1)}{dn_1^2} = -\frac{1}{n_1} - \frac{1}{N-n_1}$$

Now $B_2 = \left.\frac{d^2 \ln W(n_1)}{dn_1^2}\right|_{n_1=\bar{n}_1} = -\frac{1}{\bar{n}_1} - \frac{1}{N-\bar{n}_1}$

or $B_2 = -\frac{1}{Np} - \frac{1}{N-Np}$

$$= -\frac{1}{N}\left(\frac{1}{p} + \frac{1}{1-p}\right) = -\frac{1}{N}\left(\frac{1}{p} + \frac{1}{q}\right)$$

$$= \frac{-(p+q)}{Npq}$$

or $\boxed{B_2 = \frac{-1}{Npq}}$ ⟵ (14) "Here we have used $p+q=1$

$\Rightarrow$ B_2 is negative for W to be maximum which is required also

Next; see equation (8) $W(n_1) = \tilde{W} e^{-\frac{1}{2}|B_2|\eta^2}$

We can evaluate $\tilde{W}$ using normalization condition for $W(n_1)$, i.e. $\sum_{n_1=0}^{N} W(n_1) = 1$

or $\sum_{n_1=0}^{N} W(n_1) \simeq \int W(n_1)\,dn_1 = \int_{-\infty}^{+\infty} W(\bar{n}_1 + \eta)\,d\eta = 1$

↑ for large N

Now from (8)

$$\int_{-\infty}^{+\infty} \tilde{W} e^{-\frac{1}{2}|B_2|\eta^2}\,d\eta = 1$$

~~[using the identity $\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$ for $n = 0, 1, 2, \ldots$ & $a > 0$]~~

We obtain $\tilde{W}$:

H.W. Prove

$$\tilde{W}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}|B_2|\eta^2}\,d\eta = 1$$

Use the identity $\int_0^\infty x^n e^{-ax^2}dx = \begin{cases} \dfrac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\dfrac{\pi}{a}} & n=2k,\ k \text{ integer}, a>0, \\ \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{2\left(a^{\frac{n+1}{2}}\right)} & \text{for } n>-1,\ a>0 \\ \dfrac{k!}{2(a^{k+1})} & \text{for } n=2k+1,\ k \text{ integer} \end{cases}$

H.W. Prove this identity

$$\tilde{W}\sqrt{\frac{2\pi}{|B_2|}} = 1$$

or $\tilde{W} = \sqrt{\dfrac{|B_2|}{2\pi}}$

Thus $$\boxed{W(n_1) = \sqrt{\frac{|B_2|}{2\pi}}\, e^{-\frac{1}{2}|B_2|(n_1-\bar{n}_1)^2}} \quad (15)$$

Now using expressions for $|B_2|$ & $\bar{n}_1$ we obtain

$$W(n_1) = \sqrt{\frac{1}{2\pi Npq}}\, e^{-\frac{1}{2Npq}(n_1-Np)^2}$$

or $$\boxed{W(n_1) = (2\pi Npq)^{-1}\exp\left[-\frac{(n_1-Np)^2}{2Npq}\right]}$$

Since we have seen that Dispersion $\overline{(\Delta n_1)^2} = Npq$, we can write

$$\boxed{W(n_1) = \left[2\pi\overline{(\Delta n_1)^2}\right]^{-1}\exp\left[-\frac{(n_1-Np)}{2\overline{(\Delta n_1)^2}}\right]} \quad (16)$$

Behaves like Gaussian distribution.